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EDITORIAL

After 12 years of trying to serve SMSI successfully, I have, along with the support of many SMSI members, elevated the society to a new level where we now have teaching affiliations with two major government laboratories, Argonne National Laboratory and Fermi National Accelerator Laboratory; and we have good relations with Aurora University, which can grant graduate credit to teachers enrolled in our classes. Now we want to bring the Young People's Courses back to McRI as envisioned by Leon Urbain and Walter C. McCrone. This must continue into the future to inspire young minds to enquire into nature and to aspire to careers in science.

We have also entered the electronic age where members can get information about the SMSI expeditiously on-line and thus save monies which can be invested in course materials taught to elementary and high school students and teachers alike.

After this time span I want the society to flourish with new ideas provided by a younger person capable of leadership and scientific prowess in microscoscopy, optics, chemistry, physics, and mathematics. The integration of all these fields has been done before and must continue into the future for SMSI to advance the study of science.

Bill C. Mikuska
President, State Microscopical Society of Illinois

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Bill C. Mikuska
Editor

μ·Notes is a State Microscopical Society of Illinois publication. Its purpose is to provide a form of communication between amateur and professional microscopists, to share ideas and techniques, to ask questions, to obtain answers, to express opinions, and to publish results of experiments and research. It will also provide space for members to print wanted and for-sale notices of microscopical equipment.

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Polarizers, Retarders, and Compensators: from the Celestial Sphere to the Poincaré Sphere*

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KEYWORDS

Poincaré sphere, Mueller matrix, Mueller calculus, polarized light, retarders, compensators

INTRODUCTION AND HISTORY

The discovery of double refraction in a crystal of iceland spar (calcite) by Bartholinus (1) in 1669, the discovery of the polarization of light by using two such crystals in 1690 by Huyghens (2), followed then by the interactions of polarized light with other materials eventually led to the development of optical crystallography.

Mathematical descriptions, based both on classical and modern physics to explain phenomena such as these, are complicated and cumbersome. Noting a need for understanding and describing polarized light without complicated mathematics, George G. Stokes empirically determined what is now known as the Stokes Vector in 1852 (3), which can describe various polarization types and forms. In 1892 Henri Poincaré (4) proved that all possible polarization states described by the classical polarization ellipsoid can be represented by points on a sphere, the Poincaré Sphere, and that different polarization states are easily related to each other by various phase shifters (retarders, wave plates, compensators) placed in a given optical train, if one knows their proper location and orientation on the sphere, followed by rotation of the sphere. Further contributions by both Mueller and Jones with their calculi in the 1940s extended conceptualization and computation of polarized light/matter interaction phenomena.

This paper will provide an introduction to these topics and demonstrate their benefits to the optical researcher/polarized light microscopist by considering numerous simple experimental optical trains, calculating the final polarization state that results by using the Mueller calculus, and predicting and interpreting some of these results by means of the Poincaré Sphere.

THE STOKES VECTOR

The Stokes vector is really not a vector quantity since it only describes time averaged light intensities that have only magnitude and no direction; however, in keeping with historical convention the term *vector* will be used in this paper. The Stokes vector is a 1x4 column matrix where the matrix elements, Stokes parameters, describe the intensity and polarization state of a light beam. Although Stokes used A, B, C, D for his parameters, Walker (5) adopted I, Q, U, V, and Perrin (6) and Jones (7) adopted I, M, C, S. There are other notations as well. I, M, C, S notation is used here.

$$\begin{vmatrix} I \\ M \\ C \\ S \end{vmatrix}$$

$I = \langle I \rangle$ = total intensity

$M = \langle I_0 - I_{90} \rangle$ = the difference in intensities between horizontal and vertical linearly polarized light

$C = \langle I_{+45} - I_{-45} \rangle$ = the difference in intensities between linearly polarized light components oriented at + 45° and - 45°

$S = \langle I_{rcp} - I_{lcp} \rangle$ = the difference in intensities between right circularly polarized light and left circularly polarized light

The "< >" represents a time averaged intensity, where the time interval is sufficiently long to make a practical measurement. None of the parameters M, C, or S can be larger than the first one, I. That is, all other parameters, P, must lie in the range $-1 \leq P \leq +1$, and if a beam of light is completely polarized, then $(M^2+C^2+S^2)^{1/2} = I$. One can speak of the degree of polarization, $D=(M^2+C^2+S^2)^{1/2}/I$, if $(M^2+C^2+S^2)^{1/2} < I$. Note that the equation $(M^2+C^2+S^2)^{1/2}=I$ is the equation of a sphere of radius I, and that M, C, and S are the coordinates of a point on that sphere and rep-

* Adapted from workshop co-presented with Jan Hinsch at INTER/MICRO-2001, Chicago, IL

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resent a unique polarization state that correlates to the Poincaré sphere.

For additional information on the Stokes vector, readers are directed to two authors: Shurcliff (8), who discusses that the four Stokes parameters can be operationally defined by a set of four filters interacting with a light beam and would, therefore, follow historical development; and Collett (9) for the derivation of the Stokes parameters from electromagnetic theory.

Below are examples of normalized Stokes vectors for the more familiar polarization types and forms.

Linearly polarized light, circularly polarized light, and elliptically polarized light represent different polarization types. All linearly polarized light beams, for which the vibration direction (azimuthal or orientation angle, θ) is different, are of the same type but of different form. Note that for linearly polarized light, the Stokes' parameters M and S will vary as a linear polarizer is rotated by θ degrees about its transmission axis. These rotations are specified by sine and cosine functions in the generalized Stokes vector for linearly polarized light:

$$(1 \;\; \cos 2\theta \;\; \sin 2\theta \;\; 0).$$

Right and left circularly polarized light are of opposite forms. Note that the chirality (helicity or handedness) of these two forms are mirror images of each other. Furthermore, the current convention for the representation of right circularly polarized light corresponds to the left circularly polarized light representation found in older references. Current convention for the representation of right circularly polarized light is for the propagation direction of the light away from the source be represented by the thumb of the right hand and, as the *extended* thumb is moved towards the viewer, the curled finger tips pointing toward the palm trace a right handed helix. In older references, Shurcliff (8) and others, right circularly polarized light is represented by the thumb of the right hand directed toward the light source.

THE POINCARÉ SPHERE: PART 1

Looking upwards on a clear night man sees pinpoints of light on a dome of black fabric. When early Greek astronomers placed these points onto a "crystalline sphere", the celestial sphere, they could note the movements of these points around themselves; they then could demonstrate to other astronomers what they observed. Unfortunately, models of large celestial spheres were too unwieldly to transport from place to place. Hipparchus, a second century

$$\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \textbf{Extinction}$$

$$\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \textbf{Unpolarized light of unit intensity}$$

$$\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \textbf{Horizontal linearly polarized light}$$

$$\begin{vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{vmatrix} = \textbf{Vertical linearly polarized light}$$

$$\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix} = \textbf{+45° linearly polarized light}$$

$$\begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix} = \textbf{-45° linearly polarized light}$$

$$\begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \textbf{Right circularly polarized light}$$

$$\begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \end{vmatrix} = \textbf{Left circularly polarized light}$$

BCE Greek astronomer, found an answer to this problem; he assigned what we now know as definite latitude and longitude coordinates to each point on the sphere and then used a stereographic projection technique. With these tools, the points, stars on the celestial sphere, could now be mathematically projected onto a flat surface, a map, which could be easily rolled up and transported from place to place.

Although seemingly lost for centuries, these projection techniques were rediscovered and other projection techniques invented during the European Renaissance when classical literature of ancient Rome and Greece and exploration of the Earth became fashionable.

In 1892, Henri Poincaré published a work based on similar mapping projection techniques. The coordinates of each point of the polarization ellipsoid, derived from classical electrodynamics, were mapped onto a flat surface, the complex plane. Then reversing the technique, Poincaré proceeded to map these points onto a sphere, the Poincaré Sphere. (Some readers may note that the intermediate complex mapping step could be topologically avoided. However, it was not until 1894 that Poincaré invented algebraic topology!)

With the Poincaré sphere, all points that lie on the equator represent linear polarization states, and all of these states have an ellipticity of 0. The eastern most point on the equator, by convention, represents horizontal linearly polarized light given by the Stokes vector, (1 1 0 0). At its antipode, which is located 180° opposite therefrom, vertical linear polarized light is represented by the (1 -1 0 0) Stokes vector.

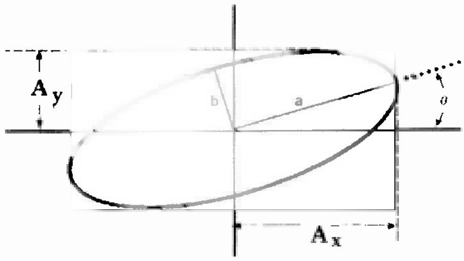
As the azimuthal angle, θ, (orientation angle) of a polarizer is rotated, there is a concomitant rotation about the polar axis of the sphere. The difference in the minimum number of degrees of rotation of a polarizer between a horizontal and vertical polarization state is 90°. The *complete* rotation of a linear polarizer from horizontal to vertical to its initial horizontal orientation involves 360°. Hence, the necessity of using 2θ in the generalized Stokes vector for linearly polarized light. Therefore, to find the point located on the equator of the Poincaré sphere of a linear polarizer oriented at 45° with respect to a horizontal linear polarizer, one must take 2θ, that is multiply 2 x 45°, and rotate the sphere about its polar axis by 90° away from the horizontal linear polarization state in an anti-clockwise manner. This results in +45° linearly polarized light. A further rotation in the same direction of the polarizer by 45°, that now is 180° away from the horizontal linear polarization state, results in vertical linearly polarized light.

The North Pole of the Poincaré sphere represents left circularly polarized light and the South Pole, right circularly polarized light. This brings us back to types of polarized light.

If one considers a circle with its center located at the origin of a two-dimensional Cartesian coordinate system, the radii along the x and y axes are equivalent. As the ratio of x and y coordinates changes, the ellipticity, b/a, changes; that is, the circle is transformed into an ellipse, and finally into a line. When the line results, we have linearly polarized light; when the radii are equal, circularly polarized light; in between, elliptically polarized light. If b=1 and a=0, there is vertical linearly polarized light, etc. (Figure 1).

Ellipticity changes are represented by movement either upwards or downwards from the equator on the Poincaré sphere where the ellipticity is zero. Therefore, great circles parallel to the equator of the Poincaré sphere represent constant ellipticity, lines of latitude. *Types* of polarized light are represented by great circles of constant latitude on the sphere and have the same ellipticity.

Longitudinal lines on the Poincaré sphere are meridians of constant azimuth. As one proceeds north or south of the equator along a given line of longitude , that is, at a given azimuth, phase difference is introduced, which ranges from 0° at the equa-



Ellipicity = b / a = tan | ω |

θ = Azimuthal Angle $-90° \le \theta \le 90°$

Poincaré

$-180° \le 2\theta \le 180°$

$-90° \le 2\omega \le 90°$

Figure 1.

tor to 90° at a given pole. In addition, a given point on the Poincaré sphere and its antipode will share a phase difference of 180°; and the coordinates, M, C, and S, of these two diametrically opposing points are interrelated by an inversion center of symmetry for the same light intensity.

Lines of different longitude represent different azimuthal angles and lines of different latitude represent different ellipticities.

Retarders, also known as phase shifters, are essentially materials that will change the ellipticity of a linear polarization state as the orientation angle of their fast or slow vibration direction is varied with respect to a given polarization state. The amount of phase shift introduced is determined by the thickness and birefringence of the retarder and by the wavelength of the light used. All anisotropic substances are variable phase shifters.

Consider the following example in terms of the Poincaré sphere. A sample is orientated at +45° with respect to an incident beam of horizontal linearly polarized light such that for the wavelength of light being used and the sample's thickness and birefringence, a phase shift of 90° results. Point H in Figure 2 is the location of the point on the equator of the sphere that represents horizontal linearly polarized light; and point P, also located on the equator at 2 x 45°, is the oriented 90° phase shifter. An arc connects P to H; and if that arc is rotated about an axis that connects the center of the sphere, O, to P, in an anti-clockwise manner (a clockwise sphere rotation) by 90° (the amount of phase shift), point H is transformed into point R, which lies at the North Pole of the sphere. This point on the sphere represents left circularly polarized light (Poincaré's convention) with a zero azimuth and an ellipticity of 1. The microscopist recognizes the sample as a $\lambda/4$ plate oriented at 45° that transforms linearly polarized light into circularly polarized light (Figure 2).

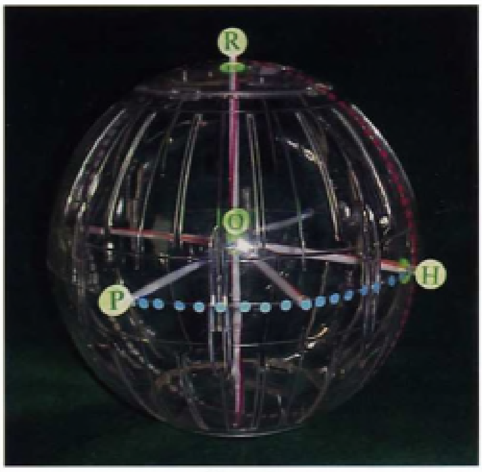


Figure 2.

Mueller Calculus

Hans Mueller of the Massachusetts Institute of Technology empirically discovered that 4 x 4 matrices, in which all matrix elements are real quantities, could be used to describe the properties of polarizers and retarders (10, 11). Furthermore, when a Stokes vector, representing some polarization state of a light beam, is left multiplied by such a matrix, the resultant 1 x 4 column matrix is the Stokes vector, which describes the new polarization state of the light beam. Shurcliff (8) discusses the basis for the development of the Mueller calculus and Parke (12, 13) demonstrates that the Mueller and the Jones calculi are interrelated, thereby showing that the Mueller calculus also has a foundation in electromagnetic theory. The Jones calculus will be very briefly addressed later.

Tables of Mueller and Jones matrices may be found in many texts on optics and polarized light. The seven Mueller matrices used in this article are given in the appendix on page 107.

Experiments and Matrix Calculations

Performing the following experiments and integrating them with the corresponding Mueller matrix calculations and Poincaré Sphere manipulations will enable the optical researcher/microscopist to grasp better the various techniques and machinations involved in the use of polarizers and compensators .

Experiment 1: Consider the interaction of unpolarized light with a single linear polarizer that has its azimuth (orientation angle) at $\theta=0°$. The Mueller matrix, M(hp), is for a horizontal linear polarizer which then left multiplies the Stokes vector for unpolarized light in the manner of linear algebra. The resultant Stokes vector describes horizontal linearly polarized light that has its intensity reduced be a factor of 1/2.

$$\mathbf{M(hp)} \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = 1/2 \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

Experiment 2: Consider a beam of unpolarized light interacting first with a linear polarizer that has its azimuthal angle at $\theta=0°$ (horizontal orientation) followed by an interaction with a linear polarizer that has its azimuthal angle at $\theta=90°$ (vertical orientation), that is, unpolarized light interacting with crossed polars. The Mueller matrix, M(hp) is for a horizontal linear polarizer and M(vp) is for a vertical linear polarizer.

$$\mathbf{M(vp)} \times \mathbf{M(hp)} \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(vp)} \times \; 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

In the last calculation the two 4 x 4 matrices, M(hp) and M(vp), were first left multiplied to give a new matrix, the null matrix, which then operated on the Stokes vector for unpolarized light. Such shortcuts are useful when similar combinations of optical elements in a given optical train are employed. The Stokes vector that results is for extinction.

Experiment 3: Consider unpolarized light interacting with three linear polarizers such that the first, second, and third polarizers have their respective azimuthal angles oriented at $\theta=0°$, $\theta=+45°$, and $\theta=90°$ respectively. This represents crossed polars with a third linear polarizer sandwiched in between, but oriented at 45°. The Mueller matrix, M(hp), is for a horizontal linear polarizer, M(45p) is for a linear polarizer oriented at 45°, and M(vp) is the Mueller matrix for a vertical linear polarizer.

We see that the introduction of a third linear polarizer at an angle of 45° allows light to be transmitted by two crossed polarizers; it is acting as a compensator. Similar phenomena are observed when optically anisotropic samples are introduced

$$\mathbf{M(vp)} \times \mathbf{M(45p)} \times \mathbf{M(hp)} \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(vp)} \times \mathbf{M(45p)} \times \; 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(vp)} \times \; 1/4 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/3 \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix} = 1/3 \begin{vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{vmatrix}$$

between crossed polars. The results are generally more complicated and the factor that precedes the final Stokes vector is different. Depending on the complexity of the interaction, especially where a knowledge of phase is of importance, the entire calculation by the Mueller matrix method is of little or no use, and the Jones calculus must be employed.

Experiment 4: Find the result of unpolarized light entering a linear polarizer oriented at θ=45° with respect to a plane mirror surface such that the reflected beam enters a second linear polarizer of the opposite vibration direction and orientation; that is, the polarizers are *crossed* and splayed against the mirror surface. Here M(m) is the matrix for a perfectly reflecting plane mirror surface, M(45p) is the matrix for a linear polarizer with the vibration direction oriented at an angle of +45° with respect to the mirror's surface, and M(-45p) is the matrix for a linear polarizer with the vibration direction oriented at an angle of -45° with respect to the mirror's surface (Figure 3).

$$\mathbf{M(-45p) \times M(m) \times M(45p) \times} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\mathbf{M(-45p) \times M(m) \times} \ 1/2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(-45p) \times} \ 1/2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 2 \\ 0 \\ -2 \\ 0 \end{vmatrix} = \begin{vmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{vmatrix} = 1/2 \begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix}$$

Note that the plane mirror changed the +45° vibration direction into a -45° vibration direction. This is equivalent to rotating one of the polarizers by 90°. Therefore, perfectly flat reflecting, non-absorbing surfaces can introduce a phase change of 180°. The chirality of left and right is interchanged; that is +y=+y upon reflection; however, +x becomes -x upon reflection.

Experiment 5: Find the result of unpolarized light entering a linear polarizer oriented at θ=45° with respect to a plane mirror surface such that the reflected beam enters a second linear polarizer of the same vibration direction and orientation; that is, the polarizers are *uncrossed* and splayed against the mirror surface. Here M(m) is the matrix for a perfectly reflecting plane mirror surface, and M(45p) used twice is the matrix for a linear polarizer oriented such that the vibration direction makes an angle of 45° with respect to the mirror's surface.

Note that the only difference between Experiment 4 and Experiment 5 is the use of a different "analyzer" orientation; therefore, a different matrix is required for the same material.

$$\mathbf{M(45p) \times M(m) \times M(45p) \times} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(45p) \times M(m) \times} \ 1/2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(45p) \times} \ 1/2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

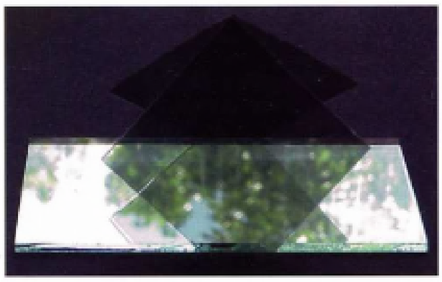


Figure 3.

Experiment 6: Find the result of unpolarized light entering a linear horizontal polarizer followed by a linear retarder oriented at θ=45° with a phase shift of δ=90°; that is, the retarder is a λ/4 plate. Refer to the earlier discussion of the Poincaré sphere. Remember, the result of using the sphere in this instance was the conversion of unpolarized light into right circularly polarized light. M(hp) is again a horizontal linear polarizer matrix and M(45, λ/4) is the matrix of a linear retarder with θ=45° and a retardance δ=90°.

Although the Poincaré sphere predicted right circularly polarized light, we see that the Mueller matrix calculation tells us that the light intensity is reduced by 1/2. This is, of course, the result of the linear horizontal polarizer and the fact that we assume a *normalized* unit intensity for the unpolarized light beam.

$$\mathbf{M(45,\lambda/4) \times M(hp)} \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{M(45,\lambda/4)} \times \ \mathbf{1/2} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} = 1/2 \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

Experiment 7: Make a sandwich combination of a linear polarizer followed by two linear retarders, each with θ=+45° and δ =90°, λ/4 plates, in coincidence (the fast and slow vibration directions of *both* must coincide), and then add a second linear polarizer, parallel to the first polarizer, against the quarter wave plate, which then interacts with unpolarized light. The Mueller matrix is M(hp) for a horizontal linear polarizer with θ=0°, and M(45, λ/4) is the matrix for a λ/4 plate. M(hp) x M(45, λ/4) x

$$\mathbf{M(45,\lambda/4) \times M(hp)} \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= \mathbf{1/2 \times M(hp)} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

The result of M(45, λ/4) x M(hp) x (1 0 0 0) produces right circularly polarized light reduced in intensity by 1/2 given by the Stokes vector 1/2 (1 0 0 1). See Experiment 6. To continue:

$$= 1/4 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Note that the inner product of M(45, λ/4) x M(45, λ/4) is the Mueller matrix for a linear retarder with θ=+45° and δ=180°, a λ/2 plate. Therefore, a λ/2 plate converts, in this instance, horizontal linearly polarized light into vertical linearly polarized light, which is why the last optical element interacting with

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

this light beam, a linear horizontal polarizer, produces extinction. This can be readily seen with the Poincaré sphere if, as in the case described in the previous section on the Poincaré sphere, the arc that connects P to H is now rotated about the axis that connects the center of the sphere, O, to P by 180°.

Experiment 8: Make a combination of a linear polarizer with a linear retarder that has θ=+45° and δ=90° (quarter wave plate), and place it onto a plane mirror such that the retarder contacts the mirror surface. The unpolarized beam of light enters the horizontal linear polarizer, interacts with the $\lambda/4$ plate with θ= +45° and δ=90°, reflects from the mirror, again interacts with the $\lambda/4$ plate, and interacts with the horizontal linear polarizer. Matrix M(hp) is used twice for the horizontal linear polarizer, and M(m) is the matrix for a plane mirror. M(45, $\lambda/4$) is the matrix for a $\lambda/4$ plate with θ=+45° and δ=90°. M(-45, $\lambda/4$) is the matrix for a $\lambda/4$ plate with θ=-45° and δ=90°. (Even though the same $\lambda/4$ plate is used in the interactions, the reflected light beam now "sees" the opposite orientation of the fast vibration direction of the retarder). The Mueller matrix calculation that describes the interaction of this system with unpolarized light is: M(hp) x M(-45, $\lambda/4$) x M(m) x

$$\text{M(45,}\lambda/4\text{) x M(hp) x} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

= 1/2 x M(hp) x

$$\text{M(-45, }\lambda/4\text{) x} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$= 1/2 \text{ x M(hp) x} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \end{vmatrix}$$

$$= 1/4 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

In the actual experiment total extinction may not be observed because the light used for this experiment is, most likely, not monochromatic, and the polarizer and retarder are not homogeneous.

Most commercial circular polarizers are made by laminating a $\lambda/4$ plate onto a linear polarizer. For these polarizers, one can determine which side of the circular polarizer is the linear polarizer and which side is the $\lambda/4$ plate by three simple means.

(1) From the results of this last experiment, a circular polarizer when placed against a plane mirror will exhibit extinction for that side of the polarizer, which is the $\lambda/4$ plate that comes in contact with the mirror.

(2) Using an additional linear polarizer, the side of the circular polarizer which exhibits extinction for certain orientations of the linear polarizer must be the linear polarizer side of the circular polarizer laminate.

(3) If a plane mirror and/or linear polarizer are not available, cut off a small fragment of the circular polarizer (this assumes an inexpensive plastic laminate) and use this fragment in one of the eight different possible orientations/configurations with respect to the circular polarizer. There will be one extinction position. This will correspond to the mutually touching sides of the fragment and circular polarizer being the linear polarizers.

Experiment 9: Make a combination of a linear horizontal polarizer sandwiched *between* two quarter wave plates with are oriented orthogonal to each other but have their transmission axes oriented at 45° with respect to the vibration direction of the linear polarizer and allow unpolarized light to interact therewith. M(45, $\lambda/4$) represents the matrix for a linear retarder with θ=45° and δ =90°, M(hp) is the matrix for a horizontal linear polarizer, and M(-45, $\lambda/4$) represents the matrix for a linear retarder with θ=-45° and δ =90°. Two cases arise:

$$M(-45,\lambda/4) \times M(hp) \times M(45,\lambda/4) \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= M(-45,\lambda/4) \times M(hp) \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= M(-45,\lambda/4) \times \ 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} = 1/2 \begin{vmatrix} 1 \\ 0 \\ 0 \\ -1 \end{vmatrix}$$

and

$$M(45,\lambda/4) \times M(hp) \times M(-45,\lambda/4) \times \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= M(45,\lambda/4) \times M(hp) \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= M(45,\lambda/4) \times \ 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$= 1/2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} = 1/2 \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

This combination of two linear retarders and one linear polarizer is an ambidextrous polarizer. That is, if unpolarized light enters from one direction, circularly polarized light results with a particular chirality; and if unpolarized light enters from the opposite direction, circularly polarized light of the opposite chirality results.

These results may be interpreted by means of the Poincaré sphere. If a linear retarder with a retardance $\delta=90°$ is placed at the $+45°$ position as described previously for the Poincaré sphere, then right circularly polarized light results for an anticlockwise rotation of the sphere. If, however, the same retarder is oriented with its fast vibration direction opposite, the starting point on the sphere is now at -45°, the antipode; then left circularly polarized light results for an anti-clockwise rotation of the sphere. This is consistant with Poincaré's convention of left and right circularly polarized light. Also, note that the North and South Poles of the sphere, which represent left and right circularly polarized light respectively, are related by a 180° rotation of the sphere that corresponds to a phase difference of the same amount.

Also notable, if the same retarder were oriented at any azimuthal angle, θ, other than 0°, 90°, or ± 45°, then elliptically polarized light results.

Experiment 10: Place a *full wave* plate between *parallel* linear polarizers such that the fast vibration direction is parallel to the vibration direction of the two linear polarizers. Depending on the thickness and material used in the manufacture of this retarder, a slight green tint will be noticeable. Repeat this with two full wave plates with their fast vibration directions parallel to the uncrossed linear polarizer's vibration directions. The depth of the green tint increases. This occurs because one full order of retardation has been introduced by the addition of the second full wave plate. Compare your results with a Michel-Lévy diagram. This final experimental result may also be demonstrated by placing a full wave plate against a plane mirror on top of which is placed a linear polarizer, which has the same vibration direction as the fast direction of the full wave plate.

Although the Mueller calculus will predict that polarized light is extant, it cannot explain the interference colors caused by phase differences; for an understanding of these phenomena one must defer to the Jones calculus.

THE POINCARÉ SPHERE: PART 2

We are now in a position to understand how anisotropic materials interact with polarized light and how compensators, such as the Berek and the de Sénarmont operate to enable the polarized light microscopist to gain optical information.

When an anisotropic material is viewed between crossed polars, two extreme possibilities eventuate, either extinction or the brightest transmission of polarized light for a given thickness and birefringence of the sample and wavelength. Of course, there is an infinitude of other possibilities. The polarized light microscopist takes advantage of one of these two extremes by rotating the sample to an extinction position. This means that one of the vibration directions of the sample must be parallel to either the vibration direction of the polarizer or the analyzer; any other orientation will result in transmission of polarized light consistent with the above restrictions.

When the microscopist then rotates the sample by 45° with respect to the extinction orientation, the sample then acts as a retarder with $\theta = \pm 45°$ and $\delta = ?°$. The retardance, δ, is dependent on the wavelength of the light used, as well as the birefringence and thickness of the sample. Under these conditions all possible retardances are points on a great circle of the Poincaré sphere. This circle is perpendicular to the equator *and* perpendicular to the rotation axis formed by the center of the sphere and one of the two possible coordinates on the equator designated by the azimuthal angle of the retarder, specifically ±45°. This great circle passes through both the North and South Poles that represent circularly polarized light as well as the two points on the equator that represent horizontal and vertical lineraly polarized light. (Figure 4).

Compensators

In their simplest sense, the compensators of the microscopist will move a point from the great circle of the Poincaré sphere to an extinction position or to a full transmission position. In practice, since the human eye cannot remember the brightest or darkest conditions, compensator manufacturers rely either on a double rotation or a half-shadow technique (14). This complication need not concern us here.

There are an infinitude of ways to bring a point from the great circle so described into an extinction or maximum brightness position; however, there are two ways which are most expeditious.

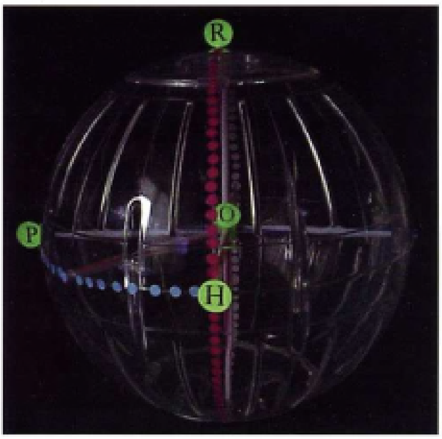


Figure 4

The simplest method to bring about this transformation is to place a compensator (variable retarder) into the light path. By tilting an oriented calcite crystal, the Berek compensator, or by sliding wedges of quartz against each other, the Babinet compensator, and others such as the Soleil and Ehringhaus, the total optical system is, ignoring human eye sensitivity problems, brought to extinction or brightest transmission. Having been calibrated with a phase shifter of known retardance, the number of degrees of rotation of the compensator's micrometer may be translated into a birefringence if the wavelength of light and thickness of the sample are known. The essence of this is exemplified by merely rotating the Poincaré sphere around the axis, formed by the center of the sphere and a point on the equator represented by the +45° or -45° equitorial positions, by the number of degrees necessary to reach a horizontal or vertical linear polarization state along the great circle.

The second method is that of de Sénarmont. This compensator has a $\lambda/4$ plate attached to it such that its fast vibration direction is parallel to the vibration direction of the polarizer. Therefore, any point on the great circle is transformed into a linear polarization state of unknown azimuth; that is, the resultant point must lie on the equator of the Poincaré sphere.

By rotating the micrometer screw of the de Sénarmont compensator's analyzer that has been calibrated against a material of known retardance, the birefringence of the sample can be determined. As the analyzer of the de Sénarmont compensator is rotated, there is a concomitant rotation of the Poincaré sphere about its polar axis. This rotation brings the sample to a full extinction or full brightness condition that corresponds to the vertical or horizontal linear polarizer position respectively.

CONCLUSION

The interaction of light with matter is a complex phenomenon. Polarized light offers clues as to what is taking place. In some instances the Mueller calculus is useful to describe the observed phenomena and makes correct predictions. In other situations, such as Experiment 10 and in a paper on the Herzog effect (15), the Jones calculus is better suited. Although this paper does not address the Jones calculus explicity, the reader should be apprised of the differences in applicability between the Jones (JC) and Mueller calculi (MC).

Appendix: Mueller Matrices

Horizontal Linear Polarizer, $\theta = 0°$

$$M(hp) = 1/2 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Vertical Linear Polarizer, $\theta = 90°$

$$M(vp) = 1/2 \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Linear Polarizer, $\theta = 45°$

$$M(45p) = 1/2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Linear Polarizer, $\theta = 45°$

$$M(-45p) = 1/2 \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Linear Retarder, $\theta = 45°$, $\delta = 90°$

$$M(45, \lambda/4) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

Linear Retarder, $\theta = 45°$, $\delta = 90°$

$$M(-45, \lambda/4) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

Mirror (linear retarder), $\theta = 0$ or $\pm 90°$, $\delta = 180°$

$$M(m) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

θ = azimuthal angle of fast transmission axis

δ = retardance

Mueller Calculas vs. Jones Calculas

1. MC can handle depolarization problems whereas JC cannot. Unpolarized or partially polarized light cannot be specified by JC matrices since all matrix elements must be orthogonal.

2. MC is phenomenological and does not depend on the validity of electromagnetic theory whereas JC is derived therefrom (however, see references 12, 13).

3. JC retains phase information, if non-normalized matrices are used, whereas MC cannot handle phase.

4. JC is well-suited to problems dealing with combining coherent beams whereas MC may be able to do so with great difficulty.

5. JC is predicated on amplitude transmittance whereas MC is predicated on intensity transmittance.

6. MC employs the Stokes vector with the first term specifying intensity whereas JC vectors do not do this; therefore, the sum of the squares of the matrix elements must be taken to obtain this information.

7. JC is well suited to determine the explicit outcome of a series or train of similar optical devices.

8. JC matrices contain 4 elements comprising 8 linearly independent constants for polarizers and retarders; that is, there is no redundant information. The MC matrices contain 16 elements and only 7 are independent. The others are redundant.

9. Many of the JC matrix elements are complex whereas all of the MC matrix elements are real.

10. Both calculi have problems handling non-linear optical devices.

MATERIALS

Linear polarizers, full wave and $\lambda/4$ plates may be purchased from the following sources:

American Polarizers, Inc., 141 South Seventh Street, Reading, PA 19602, PHONEs: (610) 373-5177 or (800) 377-4100; FAX: (610) 373-2229; or website:< www.api-optics.com >

Edmund Industrial Optics, Sales Department W011, 101 East Gloucester Pike, Barrington, NJ, 08007-1380, PHONES: (800) 363-1992 or (856) 573-6250; FAX: (856) 573-6295; E-Mail: < www.sales@edmundoptics.com >

McCrone Microscopes and Accessories, 850 Pasquinelli Drive, Westmont, IL, PHONES: (630) 887-7100, or (800) 622-8122 FAX: (630) 887-7764; or website: < www.mccrone.com >

The model of the Poincaré sphere with the 3-D Cartesian coordinate system contained inside, Figures 2 and 4, is constructed from a small animal plastic "exercise ball", a drilled plastic die, and stiff plastic rods or tubes.

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